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1993 J. Phys. A: Math. Gen. 26 3291

(http://iopscience.iop.org/0305-4470/26/13/031)

J. Phys. A: Math. Gen. 26 (1993) 3291–3299. Printed in the UK

# Relativistic kinematics of interferometry

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Received 22 October 1992

**Abstract.** By emphasizing a common kinematic core at the expense of the specific dynamics appropriate to individual configurations a unified view is obtained of a large part of the field of interferometry whether using photons or material particles. The only restriction on the coordinate system is that the metric tensor is independent of time. Within this framework a comparison is made of the main features of various types of interferometer, and the effect of dispersion is calculated for the type based on the ring laser oscillator. For a ring interferometer using material particles the particle proper time for a complete transit around the ring is shown to be the same for both directions of travel and a paradoxical consequence is noticed.

## 1. Introduction

Interferometry using light has long played an important role in elucidating fundamental aspects of physics in addition to providing useful tools in instrument technology. Recent years have seen the extension from light to material particles, with a corresponding widening of the field of fundamental aspects explored. Broadly speaking the dynamics of light interferometry is treated in terms of the electromagnetic equations and that of material particles is treated through a quantum mechanical wave equation, each experimental arrangement being tackled afresh through an appropriately chosen dynamical theory.

Although this degree of specialization is clearly needed if the fine detail is to be accommodated it is less useful in enabling the leading features of the various configurations to be readily distinguished and compared. For this purpose one seeks some common core and it is the theme of the present article that this can be achieved by emphasizing the kinematics at the expense of the dynamics. Of course the dynamics cannot be completely avoided but certainly such specific details as are provided by Maxwell's electromagnetic equations or the quantum mechanical wave equations are not needed in the development although their consequences are useful for comparison purposes.

The leading role accorded to the kinematics is reflected in the fact that most of the development rests on the central result expressing the reciprocal of the coordinate speed of a photon or material particle in terms of an index of refraction. The only restriction on the coordinate system is that the metric tensor is independent of time. Hence a wide range of gravitational effects and non-inertial motion of the interferometer can be discussed.

From an extensive literature a useful review is given by Stedman (1985), certain quantum mechanical aspects are discussed by Dieks and Nienhuis (1990) and a wide ranging view is taken by Anandan (1981).

  

It may be remarked that the initial motivation for the present article arose from a desire to confirm by other means the conclusion of an approximate electromagnetic calculation (Scorgie 1990) regarding the effects of non-inertial motion of an observer on the optical length of a given path in his 3-space. The perhaps surprising but widely accepted conclusion is that the effect of translational acceleration depends on the refractive index of matter traversed by the light and at rest in the observer's coordinate system whereas the effect of rotational motion is independent of the refractive index. The kinematic treatment readily shows how this arises as a first approximation.

Two points of detail are relegated to the appendix. The first justifies equation (3) and the second remarks on the use of the concept of refractive index for the motion of a material particle.

## 2. Kinematic relations

The setting is that an observer in arbitrary motion, and so using a non-inertial coordinate system, chooses his proper time $T$ as coordinate time. The path of an entity in 3-space is supposed known and the primary objective is to calculate the coordinate time taken to traverse the path. Of course, interference is a wave phenomenon depending on phase difference rather than time difference; consequently a frequency is involved as will be discussed later.

The term entity is intended to be non-committal: all that is required is that it should be possible to associate with it a proper time $\tau$ which may of course be zero. The physics that distinguishes one entity from another is to be introduced in terms of a refractive index defined by analogy with light. Thus the problem is to find the coordinate speed in terms of the refractive index.

The coordinates are $x^\alpha$ and $x^4=cT$ with $c$ the vacuum speed of light and $T$ the time. Greek indices run from 1 to 4, Latin indices run from 1 to 3, and repeated indices indicate summation. The square of the element of interval is

$$\mathrm{d}s^2=g_{mn}\,\mathrm{d}x^m\,\mathrm{d}x^n+2g_{m4}\,\mathrm{d}x^m\,\mathrm{d}x^4+g_{44}(\mathrm{d}x^4)^2. \tag{1}$$

The metric tensor is independent of time and $g_{44}$ is negative. In the 3-space the square of the element of coordinate length is $\mathrm{d}\sigma^2=g_{mn}\,\mathrm{d}x^m\,\mathrm{d}x^n$ leading to unit tangent $\lambda^m=\mathrm{d}x^m/\mathrm{d}\sigma$ to the path of an entity having coordinate velocity $u^m=u\lambda^m$, the coordinate speed being $u=\mathrm{d}\sigma/\mathrm{d}T$. With $\tau$ denoting proper time of the entity, (1) gives

$$-(c\,\mathrm{d}\tau/\mathrm{d}T)^2=u^2+2cg_{m4}\lambda^m u+c^2g_{44}. \tag{2}$$

In the inertial frame that is momentarily comoving with the coordinate point occupied by the entity let $t$ be the time and $\mathrm{d}p$ be the element of distance. Then what we may call the local inertial speed of the entity is $v=\mathrm{d}p/\mathrm{d}t$ and we have the usual relation

$$\mathrm{d}t/\mathrm{d}\tau=[1-(v/c)^2]^{-1/2}$$

together with

$$\mathrm{d}p^2=\gamma_{mn}\,\mathrm{d}x^m\,\mathrm{d}x^n \qquad \gamma_{mn}=g_{mn}-g_{44}^{-1}g_{m4}g_{n4}. \tag{3}$$

By analogy with light it is convenient to define the refractive index $n=c/v$, giving $\mathrm{d}t/\mathrm{d}\tau=n(n^2-1)^{-1/2}$. Then from (3)

$$\begin{aligned}(\mathrm{d}\tau/\mathrm{d}T)^2&=(\mathrm{d}\tau/\mathrm{d}t)^2(\mathrm{d}t/\mathrm{d}p)^2(\mathrm{d}p/\mathrm{d}T)^2\\&=(n^2-1)[1-g_{44}^{-1}(g_{m4}\lambda^m)^2](u/c)^2.\end{aligned} \tag{4}$$

Substituting in the left-hand side of (2) gives an equation for the coordinate speed, or rather it is preferable to find an equation for the reciprocal speed $\mu$, namely

$$c^2 g_{44}\mu^2 + 2cg_{m4}\lambda^m\mu + n^2 - (n^2-1)g_{44}^{-1}(g_{m4}\lambda^m)^2 = 0. \tag{5}$$

Since $g_{44}$ is negative

$$\mu = |cg_{44}|^{-1}\{g_{m4}\lambda^m \pm n[|g_{44}| + (g_{m4}\lambda^m)^2]^{1/2}\}. \tag{6}$$

This is indeed the central result and the remainder of the paper consists of applications. Some general features may be noted at this point. The fact that reciprocal coordinate speed is expressible in simple closed form means that the coordinate time of transit of an entity over a given path in the observer's 3-space

$$\Delta T = \int \mu \, \mathrm{d}\sigma$$

is readily found, and from the structure of (6) the mode of contribution of various influences is evident. Of particular significance is the occurrence of the refractive index not as a factor of the entire expression but as a factor of only part of it.

The concept of refractive index as a point function is natural for light since its speed at any point is determined solely by conditions there. This cannot be true of a material particle since its speed depends on its history, and this aspect is discussed in the appendix.

## 3. Aspects of interferometry

Equation (6) is exact and is needed in pursuing detail, but for a broader view an approximation is useful. For an observer in flat spacetime, having acceleration $\boldsymbol{f}$ and using space axes rotating at angular velocity $\boldsymbol{\Omega}$ with respect to his local inertial frame, we have (Scorgie 1990)

$$\boldsymbol{h} = (g_{14}, g_{24}, g_{34}) = \boldsymbol{\Omega} \times \boldsymbol{r}/c \tag{7}$$

the position vector in his 3-space being $\boldsymbol{r}$.

$$(\lambda^1, \lambda^2, \lambda^3) = \partial \boldsymbol{r}/\mathrm{d}\sigma \tag{8}$$

notation $\partial$ denoting differentiation in which the basis vectors are held constant.

$$g_{44} = -[(1 + \boldsymbol{r}\cdot\boldsymbol{f}/c^2)^2 - h^2]. \tag{9}$$

The principal effect of a weak gravitational field is to add to the right-hand side of (9) the term $2\psi/c^2$, the (positive) gravitational potential being $\psi$, the value at the observer being taken as zero. Then to first order of small quantities

$$|g_{44}|^{-1} = 1 + 2\delta \qquad \delta = (\psi - \boldsymbol{r}\cdot\boldsymbol{f})/c^2 \tag{10}$$

and (6) gives

$$\mu c = \boldsymbol{h}\cdot\partial\boldsymbol{r}/\mathrm{d}\sigma \pm n(1+\delta) \qquad \mu_+ > 0 \qquad \mu_- < 0. \tag{11}$$

This equation readily accounts for the leading aspects of interferometry in the presence of gravity and other non-inertial influences. Rotation of the observer's space axes appears only in the first term in the right-hand side, and that term is independent of the refractive index. The observer's translational acceleration, together with gravity, appears only in the second term which also contains the index of refraction as a factor.

For example it is now obvious that the previously mentioned conclusion regarding the effects of non-inertial motion on optical path length is confirmed and found to originate in kinematics rather than being peculiar to electromagnetism.

A feature of particle interferometers is their extremely high refractive index compared with that of light: values of order $10^5$ are typical for slow neutrons. Hence particle interferometers have a marked advantage over those using light in experiments exploiting gravity or translational acceleration. On the other hand, at least on the basis of present considerations, there is nothing to choose between them if rotational effects are to be explored.

But of course reciprocal speed is not the only quantity determining interference; since it is a wave phenomenon a frequency is needed to relate transit time, determined by reciprocal speed, to the phase of a wave. Thus for angular frequency $\omega$ the phase increment is

$$\varphi = \int \omega\mu \, \mathrm{d}\sigma \tag{12}$$

evaluated along the path in 3-space. Since the entire treatment is relativistic we associate with a particle of rest mass $m$ the angular frequency

$$\omega = (mc^2/\hbar)[1-(v/c)^2]^{-1/2} = n(n^2-1)^{-1/2}mc^2/\hbar. \tag{13}$$

Since the refractive index for a material particle is very large $\omega \approx mc^2/\hbar$ in (12). Consequently on a frequency basis particles are favoured over photons in the ratio particle rest energy to photon energy.

However, even if both factors in (12) favour particles they are at a disadvantage when achievable path lengths are considered. A few centimetres may be typical for particles, whereas metres or even kilometres can be contemplated for light.

Rotational effects exploit the first term on the right-hand side of (11) and the configuration is a closed ring around which a pair of similar entities travel in opposite directions, interference being detected on their return to the common launch point.

The transit times are

$$T_+ = \oint \mu_+ \, \mathrm{d}\sigma > 0 \qquad T_- = -\oint \mu_- \, \mathrm{d}\sigma > 0 \tag{14}$$

and if the refractive index is a function of position alone the phase difference is

$$\Delta = \oint \omega(\mu_+ + \mu_-) \, \mathrm{d}\sigma = 4(\omega/c^2)\boldsymbol{A} \cdot \boldsymbol{\Omega} \tag{15}$$

the vector area of an open surface spanning the ring being $\boldsymbol{A}$. This is of course the customary Sagnac approximation and is independent of the refractive index.

On the other hand gravitational and translational acceleration effects exploit the second term on the right-hand side of (11) and the open configuration is used. The two entities are launched simultaneously from a common point and travel on separate paths to another common point where interference in observed. The two limbs together from a closed ring and we have $\mu_+$ on each, subscripts 1 and 2 distinguishing one limb from the other. The transit times are

$$T_1 = \int \mu_1 \, \mathrm{d}\sigma_1 > 0 \qquad T_2 = \int \mu_2 \, \mathrm{d}\sigma_2 > 0.$$

$$c(T_1 - T_2) = \oint \boldsymbol{h} \cdot \partial \boldsymbol{r} + \int n_1(1+\delta_1) \, \mathrm{d}\sigma_1 - \int n_2(1+\delta_2) \, \mathrm{d}\sigma_2 \tag{16}$$

the first integral being taken round the closed ring formed by the two limbs and the other integrals being taken along their respective limbs. For simplicity of illustration suppose that the refractive index and gravitational potential are constant on each limb although the values may differ between the two limbs.

$$c(T_1-T_2)=(2/c)\boldsymbol{\Omega}\cdot\boldsymbol{A}+(n_1\sigma_1-n_2\sigma_2)+(n_1\sigma_1\psi_1-n_2\sigma_2\psi_2)/c^2 \\ -(n_1\sigma_1\boldsymbol{\rho}_1-n_2\sigma_2\boldsymbol{\rho}_2)\cdot\boldsymbol{f}/c^2 \tag{17}$$

the position vector of the centroid of a limb being $\boldsymbol{\rho}$.

Suppose

$$n_1\sigma_1-n_2\sigma_2=\varepsilon n_1\sigma_1 \qquad |\varepsilon|\ll 1. \tag{18}$$

$$c(T_1-T_2)\approx(2/c)\boldsymbol{\Omega}\cdot\boldsymbol{A}+n_1\sigma_1\{\varepsilon+c^{-2}[\psi_1-\psi_2-(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)\cdot\boldsymbol{f}]\}.$$

The phase difference is

$$\Delta=\Delta(\boldsymbol{\Omega})+\Delta(\sigma)+\Delta(\psi)+\Delta(\boldsymbol{f}). \tag{19}$$

The contribution from rotation is

$$\Delta(\boldsymbol{\Omega})=2(\omega/c^2)\boldsymbol{\Omega}\cdot\boldsymbol{A} \tag{20}$$

and as might be expected this is half of the corresponding contribution in the closed ring configuration. The contribution arising from imbalance between two limbs is

$$\Delta(\sigma)=\varepsilon n_1\sigma_1\omega/c. \tag{21}$$

The gravitational contribution is

$$\Delta(\psi)=n_1\sigma_1\omega(\psi_1-\psi_2)/c^3. \tag{22}$$

The contribution from translational acceleration is

$$\Delta(\boldsymbol{f})=-n_1\sigma_1\omega(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)\cdot\boldsymbol{f}/c^3. \tag{23}$$

Since the objective of this configuration is to explore the effects of gravity and translational acceleration the other two contributions are unwelcome. Allowance could be made for the rotational term (caused by Earth's rotation perhaps) by measuring it by a ring interferometer, but presumably we can only try to reduce the imbalance contribution as far as possible. Hence figures of merit are

$$\frac{\Delta(\psi)}{\Delta(\sigma)}=\frac{\psi_1-\psi_2}{\varepsilon c^2} \qquad \frac{\Delta(\boldsymbol{f})}{\Delta(\sigma)}=\frac{(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)\cdot\boldsymbol{f}}{\varepsilon c^2}. \tag{24}$$

Earth's gravitational field of strength $g$ provides illustrative numbers. Suppose the difference of gravitational potential arises from a difference of height $z$, giving

$$\Delta(\psi)/\Delta(\sigma)=gz/\varepsilon c^2\sim(z/\varepsilon)\times10^{-18} \tag{25}$$

for $z$ in centimetres. In a particle interferometer dimensions are likely to be a few centimetres, which implies a very poor figure of merit for gravitational measurements. Nevertheless such experiments have been successful (Colella *et al* 1975). It is perhaps significant that fringe shifts were detected as the plane of the two limbs was rotated about a horizontal axis, since the imbalance contribution is likely to remain substantially unaltered, leaving the gravitational contribution to be detected.

The same numbers apply if we consider using the two-limb interferometer to measure translational acceleration of a magnitude equal to that of the Earth's field at the surface. In that case $z$ becomes the projection, parallel to the acceleration, of the vector joining the centroids of the two limbs. Again a very poor figure of merit results from limb imbalance. Nevertheless proposals have been made (Clauser 1988) to measure both angular velocity and translational acceleration by means of six particle interferometers, one for each component of the two vectors.

## 4. Dispersion in a ring laser oscillator

In connection with (15) we have made the proviso 'if the refractive index is a function of position alone'. This condition will not be satisfied in the ring laser oscillator if the material traversed by the two oppositely travelling beams is dispersive since their frequencies differ slightly. In obvious notation, with $T_+$ and $T_-$ both positive, let

$$\omega_+ - \omega_- = \Delta\omega \qquad \omega = \tfrac{1}{2}(\omega_+ + \omega_-) \qquad T_+ - T_- = \Delta T. \tag{26}$$

The operating condition is $\omega_+ T_+ = \omega_- T_-$ which gives

$$\frac{\Delta\omega}{\omega} = -\frac{2\Delta T}{T_+ + T_-}. \tag{27}$$

The transit times can be found from (11), and defining

$$\Delta(n) = n_+ - n_- \qquad n = \tfrac{1}{2}(n_+ + n_-) \qquad \Delta = \sigma^{-1} \oint \Delta(n)\,\mathrm{d}\sigma \qquad N = \sigma^{-1} \oint n\,\mathrm{d}\sigma$$

the integrals being evaluated around the ring, gives

$$\frac{\Delta\omega}{\omega} = -\frac{(4/c)\boldsymbol{\Omega} \cdot \boldsymbol{A} + \oint \Delta(n)(1+\delta)\,\mathrm{d}\sigma}{\oint n(1+\delta)\,\mathrm{d}\sigma} \tag{29}$$

the vector area closed by the ring being $\boldsymbol{A}$.

Neglecting gravity and translational acceleration gives

$$\frac{\Delta\omega}{\omega} = -\frac{1}{N}[(4/\sigma c)\boldsymbol{\Omega} \cdot \boldsymbol{A} + \Delta]. \tag{30}$$

In words: $N$ is the mean value of the refractive index around the ring and $\Delta$ is the mean value, around the ring, of the difference between the two values of refractive index.

To illustrate relative magnitudes suppose the angular velocity of the Earth, $7.29 \times 10^{-5}\ \mathrm{s}^{-1}$ is to be measured with an oscillator having a path of, say, 4 m, and an enclosed area of $1\ \mathrm{m}^2$. Then the first term in brackets in (30) is about $2 \times 10^{-13}$. This indicates the sensitivity of the measurement to dispersion in the material traversed by the beams.

Notice that (29) and (30) are implicit equations to determine the frequency difference because the difference between the refractive indices for the two beams is itself a function of the frequency difference. This purely computational point can obviously be handled by an iterative procedure. At its simplest, retaining only the first order term in a Taylor expansion of the refractive index about the mid frequency, an explicit equation could be written.

It is interesting to note that the denominator of (29) shows the dependence of the frequency difference on gravity and acceleration. In the absence of dispersion

$$\frac{\Delta\omega}{\omega} = -[1-(\psi_0-\boldsymbol{\rho}\cdot\boldsymbol{f})/c^2]\frac{4\boldsymbol{\Omega}\cdot\boldsymbol{A}}{n\sigma c} \tag{31}$$

the mean gravitational potential around the ring being $\psi_0$ and the position vector of its centroid being $\boldsymbol{\rho}$. Of course these effects are minute: if the acceleration has magnitude equal to Earth's surface gravitational acceleration then $f/c^2$ is about one reciprocal light year.

## 5. Discussion

The core of the kinematic approach is the calculation of a difference between transit times whereas interference is a wave phenomenon arising from a difference between phases; hence a frequency has to be introduced to relate phase difference to time difference. For light the choice of frequency raises no qualms. For material particles we should presumably choose the frequency of the de Broglie matter wave. In the literature de Broglie's name is attached to two types of wave exhibiting large differences in some aspects although sharing some common features. The waves are typified by *free-particle solutions of the Klein–Gordon equation and the Schrödinger equation* respectively. They share the same group velocity (the velocity of the particle), virtually the same wavelength for low velocity particles, but differ greatly in frequency and phase velocity. Since our treatment is relativistic the high frequency wave has been used. An interesting account (in English) based on de Broglie's 1924 thesis is given by Dugas (1955, Part 5, Chapter 4) and this makes clear that the original approach was essentially relativistic and used the high frequency wave. As Dieks and Nienhuis (1990) discuss, there are some subtle issues here.

An aspect of material particle interferometry is worth remarking. In a closed ring configuration an observer fixed to a point on the ring launches two streams of particles in opposite directions around the ring and interference is detected at the launch point when the streams return there. For these conditions (4) gives, for one transit around the ring, the increment of particle proper time

$$\Delta\tau = c^{-1}\oint (n^2-1)^{1/2}[1-g_{44}^{-1}(g_{m4}\lambda^m)^2]^{1/2}\,\mathrm{d}\sigma. \tag{32}$$

The interesting feature is that, provided the refractive index (if not constant) is a function only of position on the ring, then the increment of particle proper time is the same for both directions of travel around a closed path of any shape. Certainly this is obvious from elementary kinematics in the case of a circular ring rotating uniformly about the perpendicular axis through its centre which remains at rest in an inertial frame. It may be less well known in the more general situation contemplated here.

Indeed this equality of particle proper times has a paradoxical aspect that may be made evident in the following way (a comma and a semicolon denote partial and covariant differentiation respectively). From

$$-c^2\,\mathrm{d}\tau^2 = g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta \qquad -c^2\,\mathrm{d}\tau = g_{\alpha\beta}\nu^\alpha\,\mathrm{d}x^\beta$$

the particle 4-velocity being $\nu^\alpha$. Starting at a fixed point on the particle's worldline, this relation may be integrated to give the increment of proper time $\tau$ as a function of the spacetime coordinates of the particle.

Hence $\tau_{,\beta}=-c^{-2}\nu_\beta$. Since $\nu_\beta\nu^\beta=-c^2$

$$g^{\alpha\beta}\tau_{,\alpha}\tau_{,\beta}=-c^{-2}. \tag{34}$$

Introducing quantum mechanics, suppose the particle wavefunction satisfies the Klein–Gordon equation for the rest mass $m$

$$g^{\alpha\beta}\psi_{,\alpha;\beta}=(mc/\hbar)^2\psi \tag{35}$$

and look for a solution $\psi=A\,\mathrm{e}^{\mathrm{i}\varphi}$. Then

$$g^{\alpha\beta}(A_{,\alpha;\beta}-A\varphi_{,\alpha}\varphi_{,\beta})=(mc/\hbar)^2A \tag{36}$$

$$g^{\alpha\beta}(A_{,\alpha}\varphi_{,\beta}+\tfrac{1}{2}A\varphi_{,\alpha;\beta})=0. \tag{37}$$

Adopting the geometrical optics approximation that derivatives of $A$ can be ignored in (36) gives

$$g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}=-(mc/\hbar)^2. \tag{38}$$

Comparison of (38) and (34) suggests the identification $\varphi=(mc^2/\hbar)\tau$. Indeed this identification is well known and has been employed for example by Stodolsky (1979) to obtain useful approximate results in a semiclassical treatment assuming the particle to travel on the exactly defined classical path.

Combining this identification with the equality of particle proper time increments previously established by (32) for the closed ring interferometer, we would deduce equality of quantum mechanical phase increments for the two oppositely travelling particle streams. This in turn would paradoxically imply that they interfere constructively and independently of gravity and non-inertial motion of the interferometer. It seems that identification of quantum mechanical phase with proper time on the classical path of a particle has to be handled with some care. For photons, of course, this feature is absent since the proper time is zero and the pair of equations (34) and (38) is replaced by the single one which is (38) with the right-hand side replaced by zero.

Although the present purpose has been to display the broad aspects at the expense of the precise detail, it is worth emphasizing that the central result (6) could be used to explore the consequences for interferometry in whatever degree of detail is desired. But of course this idea must not be pursued too far. The kinematic approach owes its simplicity to the fact that the specific dynamics is ignored. Thus it is inevitable that the dynamics of a particular situation will reveal additional features at least comparable numerically with the numbers that result from pressing (6) to high orders of accuracy. For example, particle spin has been ignored. Also in the open path arrangement each limb must obviously include a least one deflector to guide the particle even though it is in free fall for the greater part of its trajectory. In relying on the kinematic calculation one can only hope that both limbs are equally affected by the dynamics of deflection.

## Appendix

### *1. Justification for equation (3)*

It is readily confirmed that (1) may be written in the Minkowskian form

$$\mathrm{d}s^2=\mathrm{d}p^2-c^2\,\mathrm{d}t^2$$

$$\mathrm{d}p^2=(g_{mn}-g_{44}^{-1}g_{m4}g_{n4})\,\mathrm{d}x^m\,\mathrm{d}x^n$$

$$c^2\,\mathrm{d}t^2=(|g_{44}|^{1/2}\,\mathrm{d}x^4-|g_{44}|^{-1/2}g_{m4}\,\mathrm{d}x^m)^2$$

giving the interpretation that $dp$ is the element of distance and $dt$ is the element of time in an inertial system momentarily comoving with the coordinate point in question.

## 2. *Refractive index for a free particle*

Equation (6) contains the refractive index and it is not immediately clear how this concept relates to the motion of a material particle since, unlike light, its velocity depends on its history. However, for a free particle it is easy to see how the history determines the refractive index to be used in (6).

To this end let $\xi^\alpha$ be unit tangent to the geodesic traversed by the free particle. The history is most aptly described by the equation for the cotangent

$$\frac{d\xi_\alpha}{d\tau} - c\Gamma^\beta_{\alpha\gamma}\xi_\beta\xi^\gamma = 0$$

the coefficients of the connection being

$$\Gamma^\beta_{\alpha\gamma} = \tfrac{1}{2}g^{\beta\mu}(g_{\alpha\mu,\gamma} + g_{\mu\gamma,\alpha} - g_{\alpha\gamma,\mu})$$

a comma denoting partial differentiation. Then

$$\frac{d\xi_\alpha}{d\tau} = \tfrac{1}{2}c\xi^\mu\xi^\gamma g_{\mu\gamma,\alpha}$$

shows that, if the metric tensor is independent of a particular coordinate, the corresponding component of the cotangent is a constant of the motion, much as an ignorable coordinate in classical dynamics gives rise to a constant of the motion. In obvious notation

$$\xi^\alpha = (dT/c\,d\tau)(u\lambda^m, c).$$

Writing $K^{-1}$ for the constant value of $\xi_4$, since time is our ignorable coordinate, gives

$$c\,d\tau/dT = K(ug_{m4}\lambda^m + cg_{44}).$$

Substituting this relation into (2) gives an equation for the coordinate speed, or rather it is again preferable to find the reciprocal speed which takes the form of (6) with refractive index

$$n = [1 - K^2|g_{44}|]^{-1/2}.$$

The history of the particle enters through the constant $K$ which can, of course, be determined at any chosen point on the free trajectory.

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